Chapter 1

Basic Definitions and Notations

"One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than we originally put into them."

The term sequence has a great role in Analysis. Convergence of sequences has always remained a subject of interest to the mathematicians. Several new types of convergence of sequences appeared, many of them are analogous to the statistical convergence. The concept of I-convergence gives a unifying approach to such type of convergence. Statistical convergence has several applications in different fields of Mathematics, Number Theory, Trigonometric Series, Summability Theory, Probability Theory, Measure Theory, Optimization and Approximation Theory. The notion of Ideal convergence corresponds to a generalization of the statistical convergence.

Notations

$\mathbb{N} :=$ The set of all natural numbers.

$\mathbb{R} :=$ The set of all real numbers.

$\mathbb{C} :=$ The set of all complex numbers.

$\lim\limits_{k}$: means $\lim\limits_{k\to\infty}$.

$\sup\limits_{k}$: means $\sup\limits_{k\geq 1}$.

$\inf\limits_{k}$: means $\inf\limits_{k\geq 1}$, unless otherwise stated.

$\sum\limits_{k}$: means summation over $k = 1$ to $k = \infty$, unless otherwise stated.

$x := (x_k)$, the sequence whose k^{th} term is x_k.

$\theta := (0, 0, 0, ...)$, the zero sequence.

$e_k := (0, 0, .., 1, 0, 0, ..)$, the sequence whose k^{th} component is 1 and others are zeroes, for all $k \in \mathbb{N}$.

$e := (1, 1, 1, 1, ...)$.

$p := (p_k)$, the sequence of strictly positive reals.

$w := \{x = (x_k) : x_k \in \mathbb{R} \ \ (\text{or} \ \ \mathbb{C})\}$, the space of all sequences, real or complex.

$l : \{x \in w : \sum\limits_k |x_k| < \infty\}$.

$l_\infty := \{x \in w : \sup\limits_k |x_k| < \infty\}$, the space of bounded sequences.

$c_0 := \{x \in w : \lim\limits_k |x_k| = 0\}$, the space of null sequences.

$c := \{x \in w : \lim\limits_k x_k = l, \ \ \text{for some} \ \ l \in \mathbb{C}\}$, the space of convergent sequences.

l_∞, c_0, c are Banach spaces with the usual norm

$$\|x\| = \sup_k |x_k|.$$

$l_1 := \{a = (a_k) : \sum\limits_k |x_k| < \infty\}$, the space of absolutely convergent series.

$w_\infty := \{x \in w : \sup\limits_n \frac{1}{n} \sum\limits_k |x_k| < \infty\}$, the space of strongly Cesàro-bounded sequences.

$w_0 := \{x \in w : \lim\limits_n \frac{1}{n} \sum\limits_k |x_k| = 0\}$, the space of strongly Cesàro-null sequences.

$l_p := \{x \in w : \sum\limits_k |x_k|^p < \infty\}, 0 < p < \infty$.

$w_p := \{x \in w : \lim\limits_n \frac{1}{n} \sum\limits_k |x_k - l|^p = 0; \ \ \text{for some} \ \ l \in \mathbb{C}\}$.

In the case $1 \leq p < \infty$, the space l_p and w_p are Banach spaces normed by

$$\|x\| = \left(\sum_k |x_k|^p\right)^{\frac{1}{p}}$$

and

$$\|x\| = \sup\left(\frac{1}{n}\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}},$$

respectively. If $0 < p < 1$, then l_p and w_p are complete p-normed spaces, p-normed by

$$\|x\| = \sum_k |x_k|^p$$

and

$$\|x\| = \frac{1}{n}\sum_{k}^{n} |x_k|^p,$$

respectively.

The following subspaces of w were first introduced and discussed by Maddox [56] and Simons [69];

$$l(p) := \{x \in w : \sum_k |x_k|^{p_k} < \infty\}.$$

$$l_\infty(p) := \{x \in w : \sup_k |x_k|^{p_k} < \infty\}.$$

$$c(p) := \{x \in w : \lim_k |x_k - l|^{p_k} = 0, \ \text{ for some } \ l \in \mathbb{C}\}.$$

$$c_0(p) := \{x \in w : \lim_k |x_k|^{p_k} = 0\}.$$

$$w_\infty(p) := \{x \in w : \sup_k (\tfrac{1}{n} \sum_{k=1}^{n} |x_k|^{p_k}) < \infty\}.$$

$$w(p) := \{x \in w : \lim_n (\tfrac{1}{n} \sum_{k=1}^{n} |x_k - l|^{p_k}) = 0, \ \text{ for some } \ l \in \mathbb{C}\}.$$

$$w_0(p) := \{x \in w : \lim_n (\frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_k}) = 0\}.$$

Let $p = (p_k)$ be bounded. Then $c_0(p)$ is a linear metric space paranormed by:

$$g_1(x) = \sup_k |x_k|^{\frac{p_k}{M}},$$

where $M = \max(1, \sup_k p_k)$. $l_\infty(p)$ and $c(p)$ are paranormed by $g_1(x)$ defined above if and only if $\inf_k p_k > 0$. $l(p)$ and $w(p)$ are paranormed by:

$$g_2(x) = \left(\sum_k |x_k|^{p_k} \right)^{\frac{1}{M}}.$$

Remark 1.1. If $p_k = 1$, for all k, then $l_\infty(p) = l_\infty$, $c_0(p) = c_0$, $c(p) = c$, $l(p) = l$ and $w(p) = w$.

Definition 1.2. [48] A *paranorm* is a function $g : X \to \mathbb{R}$ which satisfies the following axioms: for any $x, y, x_0 \in X$, $\lambda, \lambda_0 \in \mathbb{C}$,

[i] $g(x) = 0$ if $x = \theta$;

[ii] $g(x) = g(-x)$;

[iii] $g(x + y) \leq g(x) + g(y)$;

[iv] the scalar multiplication is continuous, that is $\lambda \to \lambda_0$, $x \to x_0$ imply $\lambda x \to \lambda_0 x_0$. In other words,

$$|\lambda - \lambda_0| \to 0, \quad g(x - x_0) \to 0 \quad \text{imply} \quad g(\lambda x - \lambda_0 x_0) \to 0.$$

A paranormed space is a linear space X with a paranorm g and it is written as (X, g).

Any function g which satisfies all the conditions [i]-[iv] together with the condition

[v] $g(x) = 0$ if and only if $x = \theta$,

is called a *total paranorm* on X and the pair (X, g) is called *total paranormed space*.

Example 1.3. l_p is totally paranormed for any $p = (p_k) \in l_\infty$.

Definition 1.4. [68] Let X and Y be two nonempty subsets of the space w. Let $A = (a_{nk})$, $(n, k = 1, 2, ...)$ be an infinite matrix with elements of real or complex numbers.

We write

$$A_n(x) = \sum_k a_{nk} x_k,$$

provided the series converges. Then $Ax = (A_n(x))$ is called the *A-transform* of x.

Also

$$\lim_n Ax = \lim_{n\to\infty} A_n(x)$$

whenever it exists [68]. If $x \in X$ implies $Ax \in Y$, we say that A defines a (matrix) transformation from X into Y and we denote it by $A : X \to Y$. By (X, Y) we mean the class of matrices A that maps X into Y.

Definition 1.5. [58] A continuous function $M : \mathbb{R} \to \mathbb{R}$ is called *convex* if

$$M\left(\frac{u+v}{2}\right) \leq \frac{M(u) + M(v)}{2}, \qquad \text{for all } u, v \in \mathbb{R}.$$

If in addition, the two sides of above are not equal for $u \neq v$, then we call M to be strictly convex.

Definition 1.6. [55,58] A continuous function $M : \mathbb{R} \to \mathbb{R}$ is said to be *uniformly convex* if for any $\epsilon > 0$ and any $u_0 > 0$ there exists $\delta > 0$ such that

$$M\left(\frac{u+v}{2}\right) \leq (1-\delta)\frac{M(u)+M(v)}{2}, \qquad \text{for all } u,v \in \mathbb{R}$$

satisfying $|u-v| \geq \epsilon \max\{|u|,|v|\} \geq \epsilon u_0$.

Remark 1.7. If M is convex function and $M(0)=0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0<\lambda<1$.

Definition 1.8. An *Orlicz function* is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, nondecreasing and convex with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\to\infty$, as $x\to\infty$.

If convexity of M is replaced by $M(x+y) \leq M(x)+M(y)$, then it is called a *Modulus function*, defined and discussed by Nakano [58], Ruckle [62-64].

An Orlicz function M can always be represented in the following integral form $M(x) = \int_0^x \eta(t)dt$, where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0)=0$, $\eta(t)>0$, η is non-decreasing and $\eta(t)\to\infty$ as $t\to\infty$.

Lindenstrauss and Tzafriri [55] used the idea of Orlicz function to construct the sequence space;

$$l_M := \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\};$$

which is a Banach space with the norm

$$\|x\|_M = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}.$$

Remark 1.9. An Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0<\lambda<1.$$

For more details on Orlicz sequence spaces we refer to [55], [21-28].

Definition 1.10. An Orlicz function M is said to satisfy the $\Delta_2 - condition$ $(M \in \Delta_2$ for short$)$ if there exist constant $K \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u)$$

whenever $|u| \leq u_0$.

Definition 1.11. Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for E, a sequence space, *The Multiplier Sequence* $E(\Lambda)$, associated with the sequence Λ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. The concept of *Statistical convergence* was first introduced by Fast [12] and also independently by Schoenberg [67] for real and complex sequences.

Definition 1.12. [47] A sequence $x = (x_k)$ is called *Statistically Convergent* to L if

$$\lim_n \frac{1}{n} |\{k : |x_k - L| \geq \epsilon, k \leq n\}| = 0;$$

where the vertical bars indicate the number of elements in the set.

Remark 1.13. A sequence which converges statistically need not be convergent.

Example 1.14. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} k, & \text{if } k = n^2, \quad n \in N, \\ 0, & \text{otherwise} , \end{cases}$$

and let $L = 0$. Then

$$\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \subset \{1, 4, 9, 16, ..., k^2, ...\}.$$

We have that

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0, \ \ \text{for every} \ \ \epsilon > 0.$$

This implies that the sequence (x_k) converges statistically to zero. But the sequence (x_k) does not converge to L.

Remark 1.15. A sequence which converges statistically need not be bounded. c.f([5], [7-9], [11], [13], [21-28], [29-38], [39-46].)

Asymptotic and Logarithmic Density 1.16. If $A \subseteq \mathbb{N}$, then χ_A denotes characteristic function of the set A, i.e.

$$\chi_A(k) = 1 \quad \text{if} \ \ k \in A$$

and

$$\chi_A(k) = 0 \quad \text{if} \ \ k \in \mathbb{N} - A.$$

Put

$$d_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k),$$

$$\delta_n(A) = \frac{1}{S_n}\sum_{k=1}^{n} \frac{\chi_A(k)}{k},$$

where

$$S_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Then the numbers

$$\underline{d}(A) = \lim \inf_{n \to \infty} d_n(A),$$

$$\bar{d}(A) = \lim_{n\to\infty} \sup d_n(A),$$

are called the lower and upper asymptotic density of A, respectively(cf.[60],p.71). Similarly, the numbers

$$\underline{\delta}(A) = \lim_{n\to\infty} \inf \delta_n(A),$$

$$\bar{\delta}(A) = \lim_{n\to\infty} \sup \delta_n(A),$$

are called the lower and upper logarithmic density of A, respectively. If there exists

$$\lim_{n\to\infty} d_n(A) = d(A),$$

and

$$\lim_{n\to\infty} \delta_n(A) = \delta(A),$$

then $d(A)$ and $\delta(A)$ are called the asymptotic and logarithmic density of A respectively. It is well known fact, that for each $A \subseteq \mathbb{N}$,

$$\underline{d}(A) \leq \underline{\delta}(A) \leq \bar{\delta}(A) \leq \bar{d}(A).$$

Hence if $d(A)$ exists, then $\delta(A)$ also exists and $d(A) = \delta(A)$. The numbers $\underline{d}(A), \bar{d}(A), d(A), \underline{\delta}(A), \bar{\delta}(A), \delta(A)$ belong to the interval [0,1]. Owing to the well known formula

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = ln\, n + \gamma + 0(\frac{1}{n}), n \to \infty,$$

where γ is the Eulers constant.

Definition 1.17. Let $X \neq \phi$. A class $I \subset 2^X$ of subsets of X is said to be an *Ideal* in X if

[i] $\emptyset \in I$;

[ii] $A, B \in I$ imply $A \cup B \in I$;

[iii] $A \in I, B \subset A$ imply $B \in I$.

An ideal is called non-trivial if $X \notin I$ while an admissible ideal I further satisfies $\{x\} \in I$ for each $x \in X$.

Definition 1.18. Let $X \neq \phi$. A non-empty class $\pounds \subset 2^X$ of subsets of X is said to be *Filter* in X if

[i] $\emptyset \notin \pounds$;

[ii] $A, B \in \pounds$ imply $A \bigcap B \in \pounds$;

[iii] $A \in \pounds, B \supset A$ imply $B \in \pounds$,

The following Proposition expresses the relation between the notions of Ideals and Filters.

Proposition 1.19. Let I be a non-trivial ideal in X, $X \neq \phi$. Then the class

$$\pounds(I) = \{M \subseteq X : \exists A \in I : M = X - A\},$$

is a filter on X.

The concept of statistical convergence and the study of similar type of convergence lead to the introduction of the notion of I-convergence of sequences. The notion gives a unifying look at many types of convergence related to statistical convergence.

Definition 1.20. Let I be a non-trivial ideal in $\mathbb{N}$. A sequence $x = x_k$ of real numbers is said to be I-convergent to $\xi \in \mathbb{R}$ if for every $\epsilon > 0$ the set

$$A(\epsilon) = \{k : |x_k - \xi| \geq \epsilon\} \in I.$$

If $x = (x_k)$ is I-convergent to ξ we write $I - \lim x_k = \xi$ and the number ξ is called the I-limit of $x = (x_k)$.

The concept of I-convergence satisfies some usual axioms of convergence listed below:

[i] Every stationary sequence $x = (\xi, \xi, ...\xi, ...)$ I-converges to ξ.

[ii] The uniqueness of limit: If $I - \lim x = \xi$ and $I - \lim x = \eta$, then $\xi = \eta$.

[iii] If $I - \lim x = \xi$, then for each subsequence y of x we have $I - \lim y = \xi$.

[iv] If each subsequence y of a sequence x has a subsequence z I-convergent to ξ, then x is I-convergent to ξ.

Examples of Ideals 1.21.

[i] $I_0 = \emptyset$. This is the minimal non-empty non-trivial ideal in $\mathbb{N}$. A sequence is I_0 convergent if and only if it is constant.

[ii] Let $\phi \neq M \subseteq N$, $M \neq N$. Let$I_M = 2^M$. Then I_M is a non trivial ideal in N. A sequence $x = (x_k)$ is I_M-convergent if and only if it is constant on N-M.

[iii] Let I_f denotes the class of all finite subsets of $\mathbb{N}$. Then I_f is an admissible ideal in $\mathbb{N}$ and I_f-convergence coincides with the usual convergence in $\mathbb{R}$.

[iv] Let

$$I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}.$$

Then I_d is an admissible ideal in $\mathbb{N}$ and I_d-convergence coincides with the statistical convergence.

[v] Let

$$I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}.$$

Then I_δ is an admissible ideal in $\mathbb{N}$ and we call the I_δ-convergence the logarithmic statistical convergence.

[vi] The examples [iv] and [v] can be generalised by choosing $c_n > 0$, such that

$$\sum_{n=1}^{\infty} c_n = +\infty.$$

Putting

$$h_m(A) = \frac{\sum\limits_{i\leq m, i\in A} c_i}{\sum\limits_{i=1}^{m} c_i} \qquad \text{(m=1,2,3......)} .$$

Denote by $h(A)$ the $\lim_{m\to\infty} h_m(A)$. Then

$$I_h = \{A \subseteq \mathbb{N} : h(A) = 0\},$$

is an admissible ideal in $\mathbb{N}$ and I_d and I_δ-convergence are special cases of I_h convergence.

[vii] Let $u(A)$ denotes the uniform density of the set A. Then

$$I_u = \{A \subseteq \mathbb{N} : u(A) = 0\},$$

is an admissible ideal in $\mathbb{N}$ and I_u-convergence will be called the uniform statistical convergence.

[viii] Let $T = (t_{n,k})$ be a non-negative regular matrix, then for each $A \subseteq \mathbb{N}$ the series

$$d_T^{(n)}(A) = \sum_{k=1}^{\infty} t_{n,k}\chi_A(k) \qquad \text{(n=1,2,3......)} ,$$

converges if there exist

$$d_T(A) = \lim_{n\to\infty} d_T^{(n)}(A).$$

Then $d_T(A)$ is called the T-density of A. Putting

$$I_{d_T} = \{A \subseteq \mathbb{N} : d_T(A) = 0\},$$

then I_{d_T} is an admissible ideal in $\mathbb{N}$.

[ix] Let v be a finite additive measure defined on a class $\mathcal{U}$ of subsets of $\mathbb{N}$ which contains all finite subsets of $\mathbb{N}$ and $v(\{n\}) = 0$ for each $n \in \mathbb{N}$. $v(A) \leq v(B)$ if $A, B \in \mathcal{U}$, $A \subseteq B$. Then

$$I_v = \{A \subseteq \mathbb{N} : v(A) = 0\}$$

is an admissible ideal in $\mathbb{N}$.

[x] Let $\mu_m : 2^N \to [0, 1]$, m=1,2,.... be finitely additive measures defined on 2^N. If there exists

$$\mu(A) = \lim_{m\to\infty} \mu_m(A),$$

then $\mu(A)$ is called the measure of A, and

$$I_\mu = \{A \subseteq \mathbb{N} : \mu(A) = 0\},$$

is an admissible ideal in $\mathbb{N}$.

[xi] Let

$$N = \bigcup_{j=1}^{\infty} D_j,$$

be a decomposition of N (i.e $D_k \cap D_l = \phi$ for $k \neq l$). Assume that $D_j(j = 1, 2, ...)$ are infinite sets. Choose $D_j = \{2^{j-1}(2s - 1) : s = 1, 2...\}$. Denote by $\mathcal{J}$ the class of all $A \subseteq \mathbb{N}$ such that A intersects only a finite number of D_j. Then it is easy to see that $\mathcal{J}$ is an admissible ideal in $\mathbb{N}$.

[xii] The concept of density ρ of sets $A \subseteq \mathbb{N}$ is axiomatically introduced. Using this concept we can define the ideal

$$I_\rho = \{A \subseteq \mathbb{N} : \rho(A) = 0\},$$

and obtain I_ρ-convergence as a generalization of statistical convergence.

Relation between I-Convergence and μ-statistical Convergence 1.22.

The approach of Connor[7-9] towards the generalization of statistical convergence is based on using a finite additive measure μ defined on the field Γ of subsets of $\mathbb{N}$ with $\mu(\{k\}) = 0$ for each $k \in \mathbb{N}$ and such that $A, B \in \Gamma, A \subseteq B$ implies $\mu(A) \leq \mu(B)$. If we put

$$I = \{A \in \Gamma : \mu(A) = 0\},$$

then it is easy to verify that I is an admissible Ideal in $\mathbb{N}$ and

$$\pounds(I) = \{B \subseteq \mathbb{N} : \mu(B) = 1\}.$$

Conversely, if I is an admissible Ideal in $\mathbb{N}$, then we put

$$\Gamma = I \cup \pounds(I).$$

Then Γ is a field (Algebra) of subsets of $\mathbb{N}$. Define $\mu : \Gamma \to \{0, 1\}$ as follows:

$$\mu(M) = 0 \quad \text{if} \quad M \in I$$
$$\mu(M) = 1 \ \text{ if } \ M \in \pounds(I).$$

Now it is easy to see that $I \cap \pounds(I) = \phi$ and $\mu(\{k\}) = 0$. Also the monotonicity and additivity of μ is preserved. Hence these two approaches towards generalization of statistical convergence seem to be equivalent in such a sense that each of them can be replaced by the other.

Fundamental arithmatical properties of I-convergence 1.23.

I-Convergence has arithmatical properties similar to the properties of the usual convergence.

Theorem 1.24. Let I be a non-trivial ideal in $\mathbb{N}$

(i) If $I - \lim x_n = \xi$, $I - \lim y_n = \eta$, then $I - \lim(x_n + y_n) = \xi + \eta$.

(ii) If $I - \lim x_n = \xi$, $I - \lim y_n = \eta$, then $I - \lim(x_n.y_n) = \xi.\eta$.

(iii) If I is an admissible ideal in $\mathbb{N}$, then $\lim_{n\to\infty} x_n = \xi$ implies $I - \lim x_n = \xi$.

Definition 1.25. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$. In this case we write $I - \lim x_k = L$. The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C} \}.$$

Definition 1.26. A sequence $(x_k) \in \omega$ is said to be I-null if L = 0 . In this case we write $I - \lim x_k = 0$.

Definition 1.27. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number m = m(ϵ) such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in$ I.

Definition 1.28. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists M >0 such that $\{k \in \mathbb{N} : |x_k| > M\} \in I$.

Definition 1.29. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where K is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$.

Definition 1.30. A convergence field of I-convergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$.

Define a function $\hbar : F(I) \to \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbb{R}$ is a Lipschitz function.

Definition 1.31. A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence α_k of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

Definition 1.32. A sequence space E is said to be a sequence algebra if

$$(x_k) * (y_k) = (x_k y_k) \in E \quad \text{whenever} \quad (x_k), (y_k) \in E.$$

Definition 1.33. A sequence space E is said to be convergencefree if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Definition 1.34. A sequence space E is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$ where π is a permutation on $\mathbb{N}$.

Definition 1.35. A sequence space E is said to be monotone if it contains the canonical preimages of its step spaces. (c.f.[2], [4], [6], [10], [17], [47], [48-49], [53-54], [60-61], [65-66], [70], [71-773], [74], [74], [76]).